About nonlinear geometric optics

E. Dumas

Institut Fourier, UMR 5582 (CNRS-UJF)
100 rue des Mathématiques
Domaine Universitaire
BP 74, 38402 Saint Martin d'Hères - France
email: edumas@ujf-grenoble.fr

Abstract

We give an idea of the evolution of mathematical nonlinear geometric optics from its foundation by Lax in 1957, and present applications in various fields of mathematics and physics.

1   Introduction

Geometric optics goes back at least to the XVIIth Century, with Fermat, Snell and Descartes, who described the "paths" (rays) followed by the light. Nowadays, Physics tells us that we may reasonably replace the waves from Quantum Mechanics with classical particles, in the semi-classical approximation (when considering Planck's constant $\hbar$, or the wavelength, as infinitely small). The mathematical transcription of these problems consists in studying the asymptotic behavior of solutions to partial differential equations where different scales (represented by small parameters) are present, often in a high frequency oscillatory context.

We present the first historical results of the field, and then review some extensions and applications of the method. We shall see how geometric optics applies to Maxwell's equations (from optics, ferromagnetism, ...), to the wave or Klein-Gordon equation, to fluid dynamics and plasma physics, to general hyperbolic systems and conservation laws, as well as to nonlinear Schrödinger equations, among others. Furthermore, we shall see that it does not apply to oscillatory problems only, but also to boundary layers, shocks and long waves problems.

1

Contents

Quoting some reviews and introductory texts may be useful. A review on nonlinear geometric optics in 1998 is due to Joly, Métivier and Rauch [JMR99], and the online book of Rauch [Rau96] is a nice introduction to the subject. Majda [Maj84] made major contributions in the 80ies. Good Physics textbooks on nonlinear optics are Newell and Moloney [NM92] and Boyd [Boy03]. Kalyakin [Kal89] and Hunter [Hun95] review many questions, methods and applications about these multi-scale problems, and Whitham [Whi99] had pioneering contributions in the modeling of nonlinear waves.

2 First steps

2.1 Linear geometric optics

The first rigorous result in mathematical geometric optics is due to Lax [Lax57], who shows that strictly hyperbolic systems admit *WKB* (for Wentzel, Kramers and Brillouin) solutions. These have the *phase-amplitude* representation

$$u^\varepsilon(t,x) = a^\varepsilon(t,x) \ e^{i\varphi(t,x)/\varepsilon}.$$

The wavelength corresponds to the small parameter $\varepsilon > 0$, and the amplitude a^ε admits an asymptotic (in general not convergent) expansion,

$$a^\varepsilon \sim \sum_{n \geq 0} \varepsilon^n a_n(t,x), \quad \text{as } \varepsilon \to 0.$$

In particular, Lax uses such solutions to study the well-posedness of the Cauchy problem.

The geometric information is contained in the *eikonal equation* (of Hamilton-Jacobi type) determining the phase φ (plane waves correspond to phases which are linear in (t,x); spherical waves, to phases which are functions of t and $|x|$ only). The amplitudes a_n are solutions to *transport equations* along the rays associated with the eikonal equation. The asymptotic expansion is valid for times before rays focus.

2.2 Weakly nonlinear geometric optics

Trying to generalize Lax's approach to nonlinear systems, one immediately faces several problems. First, a family $\left(a(x)e^{i\varphi(x)/\varepsilon}\right)_{\varepsilon>0}$ is not bounded in any Sobolev space H^s with $s > 0$, so that the time of existence of the considered exact solution u^ε may shrink to zero as ε goes to zero. In the same spirit, the size of u^ε must be adapted, large enough to allow nonlinear features at first order, but sufficiently small to prevent blow-up. Finally, the main interest and difficulty of nonlinear models comes from interactions: one hopes to incorporate in the asymptotic description the self-interaction of a wave (leading for example to generation of harmonics) as well as the possibility of (resonant) interaction of several waves.

Thus, the extension to nonlinear systems goes through *profiles* $U_n(t,x,\theta)$

3

which are periodic or almost periodic w.r.t. $\theta \in \mathbb{R}^q$,

$$(1) \qquad u^\varepsilon \sim \underline{u}(t,x) + \varepsilon^m \sum_{n \geq 0} \varepsilon^n U_n\left(t, x, \vec{\varphi}(t,x)/\varepsilon\right),$$

where $\underline{u}$ is a given groundstate, and $\vec{\varphi} = (\varphi_1, \ldots, \varphi_q)$ is a collection of phases.

The usual strategy of nonlinear geometric optics consists in: 1-defining a *formal* solution, *i.e.* solvable equations for the profiles; truncating the series in (1) defines an approximate solution (a function approximately solution to the equation); 2-showing that for any initial data close to the initial value of the approximate solution, an exact solution exists on a time interval independent of ε; 3(*stability*)-showing that the exact solution is well approximated by the formal one.

In order to observe some nonlinear behavior, the magnitude ε^m of oscillations is chosen so that cumulated effects of nonlinearities become of the same order as the wave on the typical time T of propagation. A fixed T (w.r.t. ε) leads to *nonlinear geometric optics*. Once rescaled, the system of partial differential equations takes the form

$$(2) \qquad L(t, x, u^\varepsilon, \varepsilon\partial)u^\varepsilon = F(t, x, u^\varepsilon),$$

where the operator L is in general a first order symmetric hyperbolic system on some domain in $\mathbb{R}^{1+d}_{t,x}$,

$$(3) \qquad \begin{aligned} L(t, x, u, \varepsilon\partial)\ u &= \varepsilon\partial_t u + \sum_{j=1}^d A_j(t, x, u)\varepsilon\partial_{x_j} u^\varepsilon + L_0(t, x)\ u \\ &=: L_1(t, x, u, \varepsilon\partial)\ u + L_0(t, x)\ u. \end{aligned}$$

The smooth functions F, B and A_j take values in $\mathbb{C}^N$, in the space of $N \times N$ matrices, and in the space of $N \times N$ symmetric matrices, respectively. Furthermore, we will distinguish the hyperbolic case, when $L_0 = 0$, and the dispersive case, when L_0 is skew-symmetric (the dissipative case, when $L_0 + L_0^\star \geq 0$, will be considered only in Section 7). This coefficient L_0 reflects the interaction between the wave and the material medium, so that, in the dispersive case, the group velocity (see (9)) depends on the frequency. The abovementioned exponent m is then defined as follows. Let $J \geq 2$ be the order of nonlinearities,

$$|\alpha| \leq J - 2 \quad \Rightarrow \quad \partial_{u\bar{u}}^\alpha A_{j|_{u=0}} = 0, \qquad |\beta| \leq J - 1 \quad \Rightarrow \quad \partial_{u\bar{u}}^\beta F_{|_{u=0}} = 0.$$

4

The standard amplitude of weakly nonlinear geometric optics corresponds to

$$(4) \qquad\qquad m = \frac{1}{J-1}$$

(so that $m = 1$ for quadratic semi- or quasilinear systems).

As an example, Euler equations for compressible and entropic gas dynamics form a quasilinear hyperbolic system ($L_0 = 0$), and Maxwell-Bloch equations, a semilinear dispersive one (describing the propagation of an electromagnetic wave (E, B) in a medium with polarization vector P; for a two-level quantum medium, N is the difference of the populations of the energy levels), which reads

$$(5) \qquad \begin{cases} \partial_t B + \operatorname{curl} E = 0, \\ \partial_t E - \operatorname{curl} B = -\partial_t P, \\ \partial_t N = E \cdot \partial_t P, \\ \varepsilon^2 \partial_t^2 P + P = (N_0 - N)E. \end{cases}$$

It may be written in the form (2) for the unknowns $B, E, N, P, \varepsilon \partial_t P$.

2.3 Profile equations

The formal derivation of profile equations is similar in any dimension, and for any number of phases. Plug expansion (1) into equation (2), and let the expansion of $Lu - F(u)$ vanish. This gives an infinite set of equations,

$$L(\mathrm{d}\vec{\varphi} \cdot \partial_\theta)\, U_0 = 0,$$

and for $n \geq 0$,

$$(6) \qquad L(\mathrm{d}\vec{\varphi} \cdot \partial_\theta)\, U_{n+1} + L_1(\partial)\, U_n + F_n(U_0, \dots, U_n) = 0,$$

with $\mathrm{d}\vec{\varphi} \cdot \partial_\theta = \sum_{k=1}^q \mathrm{d}\varphi_k \partial_{\theta_k}$.

The analysis is based on formal Fourier series in θ, so that for $U = \sum_\alpha U^\alpha e^{i\alpha \cdot \theta}$,

$$0 = L(\mathrm{d}\vec{\varphi} \cdot \partial_\theta)\, U = \sum_\alpha L(i\mathrm{d}(\alpha \cdot \vec{\varphi}))\, U^\alpha e^{i\alpha \cdot \theta} \iff U = \Pi\, U = \sum_\alpha \pi_\alpha U^\alpha e^{i\alpha \cdot \theta},$$

where π_α is the orthogonal projector on the kernel of $L(id(\alpha \cdot \vec{\varphi}))$. In order to get non-trivial solutions, this projector is assumed not to vanish at least for one multi-index α. This means that the function $\alpha \cdot \vec{\varphi}$ satisfies the equation

$$(7) \qquad \det L(id(\alpha \cdot \vec{\varphi})) = 0,$$

so that $\alpha \cdot \vec{\varphi}$ must satisfy one of the eikonal equations,

$$\partial_t(\alpha \cdot \vec{\varphi}) + \lambda_k(t, x, \partial_x(\alpha \cdot \vec{\varphi})) = 0.$$

Here, we denote by $\lambda_k(t, x, \xi)$, $1 \leq k \leq N$, the N eigenvalues of the symmetric matrix

$$\frac{1}{i} L(t, x, 0, (0, i\xi)) = \sum_{j=1}^{d} \xi_j A_j(t, x, 0) + \frac{1}{i} L_0(t, x).$$

In particular, a single initial phase $\varphi^0(x)$ may generate N eikonal phases $\varphi_k(t, x)$ (except when initial profiles are prepared, *i.e.* polarized on a single mode).

When the linear combination $\alpha \cdot \vec{\varphi}$ is not trivial, *i.e.* contains linearly independent φ_k's, equation (7) expresses a *resonance* between these phases. Such resonances must involve at least three phases, and thus occur for systems of size at least 3.

Profile equations are then obtained recursively, splitting (6) thanks to the projectors Π and $(1 - \Pi)$,

$$(8a) \qquad (1 - \Pi)\, U_0 = 0, \qquad \text{and for } n \geq 0,$$
$$(8b) \qquad \Pi\, L_1(\partial)\, \Pi\, U_n + \Pi\, F_n = 0,$$
$$(8c) \qquad (1 - \Pi)\, U_{n+1} = -L(d\vec{\varphi} \cdot \partial_\theta)^{-1}[L_1(\partial)\, U_n + F_n].$$

Equation (8a) is a *polarization* constraint on the first profile U_0, and (8b) gives the evolution for the polarized part $\Pi\, U_n$ of U_n in the domain $\Omega \subset \mathbb{R}^{1+d}$ considered. The first equation (8b) (for the first profile U_0) is nonlinear, while the others (for $n \geq 1$) are linear. The operator $L(d\vec{\varphi}\cdot\partial_\theta)^{-1}$ is formally defined by the symbol $L(id(\alpha \cdot \vec{\varphi}))^{-1}$, the partial inverse of $L(id(\alpha \cdot \vec{\varphi}))$ on its range.

The description of the operator $\Pi\, L_1\, \Pi$ depends on the geometry of the characteristic variety,

$$\mathcal{C} = \{(t, x, \tau, \xi) \in \Omega \times \mathbb{R}^{1+d} \mid \det L(t, x, \tau, \xi) = 0\},$$

viewed as a differentiable manifold. In the hyperbolic case ($L_0 = 0$), this characteristic variety is homogeneous in (τ, ξ): for the wave equation, it is the light cone $\mathcal{C} = \Omega \times \{\tau^2 = |\xi|^2\}$. In the dispersive case ($L_0 = L_0^\star \neq 0$), only few harmonics of an eikonal (τ, ξ) are eikonal in general: for Klein-Gordon equation, $\mathcal{C} = \Omega \times \{\tau^2 = |\xi|^2 + 1\}$.

When for all (t, x) $\in \Omega$, the vector $\mathrm{d}(\alpha \cdot \vec{\varphi})(t, x)$ is a smooth point of a sheet $\tau = \lambda(\tau, \xi)$ of $\mathcal{C}$, the principal part of $\pi_\alpha L_1(\partial)\pi_\alpha$ is simply the (scalar) *transport operator* at the *group velocity*,

(9)
$$\vec{v}_\alpha(t, x) = -\frac{\partial \lambda}{\partial \xi}(t, x, \partial_x(\alpha \cdot \vec{\varphi})(t, x));$$

(for the wave equation, $\partial_\xi(\pm|\xi|) = \pm \xi/|\xi|$ does not depend on $|\xi|$, whereas for Klein-Gordon equation, $\partial_\xi(\pm\sqrt{|\xi|^2 + 1}) = \pm\xi/\sqrt{|\xi|^2 + 1}$ does). Precisely ([Lax57], [DR97a]),

$$\pi_\alpha L_1(\partial)\pi_\alpha = \partial_t + \vec{v}_\alpha \cdot \partial_x + \pi_\alpha(L_1(\partial)\pi_\alpha).$$

This reduction clearly explains the "light rays" picture; it is however not necessary in the justification of the asymptotics (1); as an example, Joly, Métivier and Rauch perform this justification [JMR94] in the case of an eigenvalue λ which changes multiplicity (crossing in the characteristic variety), which gives a model for conical refraction of light. Moreover, this description is clarified by Lannes [Lan98], in the case of isolated singular points of a dispersive $\mathcal{C}$: the characteristic variety of $\pi_\alpha L_1(\partial)\pi_\alpha$ is then the tangent cone to $\mathcal{C}$. A systematic approach to such "algebraic lemmas" is given by Texier [Tex04] *via* perturbation theory of complex eigenvalues under a slightly different smoothness assumption (found by Métivier and Rauch [MR03] to be actually the same in the hyperbolic case). This approach also generalizes the one of Lannes, and applies to "all order algebraic lemmas" (see Section 6).

2.4 Rigorous results

One easily suspects, in view of the term $L(\mathrm{d}\vec{\varphi} \cdot \partial_\theta)^{-1}$ in (8c), that resonances may cause troubles. That is the reason why monophase expansions (with profiles polarized on a single mode) have been studied first.

In [CB69], Choquet-Bruhat constructs formal such expansions for quasilinear systems, and applies these to computations on the model of relativistic

perfect fluids. Single-phase weakly nonlinear geometric optics is justified in space dimension d, for semilinear systems, by Joly and Rauch [JR92] in the non-dispersive case, by Donnat and Rauch [DR97a], [DR97b] in the dispersive case, and by Guès [Guè93] for quasilinear non-dispersive systems. Their strategy consists in constructing an approximate solution $u_{\mathrm{app}}^{\varepsilon}$, sum of a high number of smooth profiles (or with infinitely accurate asymptotics, *via* Borel's summation technique), so as to absorb the oscillations of the residual $Lu_{\mathrm{app}}^{\varepsilon} - F(u_{\mathrm{app}}^{\varepsilon})$. Existence of exact solutions u^{ε} and stability are then obtained *via* energy estimates on $u^{\varepsilon} - u_{\mathrm{app}}^{\varepsilon}$.

The formal study of propagation and interactions of oscillations for quasilinear systems, particularly for fluid dynamics, is done in the works of Hunter and Keller [HK83] in the non interacting case, of Majda and Rosales [MR84] in the one dimensional resonant case, of Hunter, Majda and Rosales [HMR86] and Pego [Peg88] in the multidimensional resonant case. In these works, a finiteness assumption is made on the number of (directions of) eikonal phases. But in [JR91], Joly and Rauch show that resonances may generate dense oscillations in the characteristic variety.

This finiteness hypothesis is shown unnecessary in [JMR93b], where Joly, Métivier and Rauch justify rigorously the weakly nonlinear geometric optics approximation in space dimension 1, for both semi- and quasilinear systems (before shock formation), under a weaker transversality assumption between real combinations s of phases and the propagation fields X_k ($X_k s \equiv 0$, or $X_k s \neq 0$ almost everywhere on the domain Ω). They use the method initiated by Joly [Jol83] for semilinear multidimensional systems with constant coefficients and linear phases, estimating (by stationary and non-stationary phase method) the difference $u_{\varepsilon}^{k} - \varepsilon^{m} U_{0}^{k}(t, x, \vec{\varphi}/\varepsilon)$ between the k-th Picard iterates obtained from the resolution of (2) and of the first profile equation, respectively. This leads to a first order approximation,

$$u^{\varepsilon}(t, x) = \varepsilon^{m} U_0(t, x, \vec{\varphi}(t, x)/\varepsilon) + o(\varepsilon^{m}),$$

which is valid in L^p for all $p < \infty$. It holds in L^{∞} when transversality is imposed (almost everywhere) along integral curves of each propagation field X_k. Guès shows [Guè95a] that on the contrary, when weak resonances are present (a linear combination of phases is eikonal on a set with one dimensional positive measure, but is a.e. not eikonal on Ω), they may lead to the creation of stationary waves preventing from L^{∞} approximation.

Concerning results in space dimension one, we have the earlier works (for linear phases) of Kalyakin [Kal89], and also of Tartar *et al.* [Tar81], [Tar84],

8

[MPT85], by the compensated compactness method applied to Young measures –links between nonlinear geometric optics and compensated compactness are very clearly enlighted in [MS98] and [JMR95c].

Furthermore, still in space dimension one, several papers deal with (global) weak solutions (with bounded variations) of conservation laws. The formal study is due to Majda and Rosales [MR84]; DiPerna and Majda obtain $L_t^\infty L_x^1$ global in time asymptotics –even when shocks occur–, using L^1 stability and BV decay properties of solutions (constructed by Glimm's scheme) to conservation laws, for BV initial data with compact support. This result is extended by Schochet [Sch94b] to periodic BV data using WKB expansions, and then by Cheverry [Che96a], [Che97] to general BV data. Sablé-Tougeron [ST96] treats the initial-boundary value problem, for linear non-interacting phases, and data with compact support.

Now, the multidimensional case for semi- or quasilinear interacting waves is treated by Joly, Métivier and Rauch [JMR95a]. In space dimension greater or equal to 2, the new phenomenon is the *focusing* of oscillations, corresponding to singularities of phases (see Section 4). It may lead to blow-up, and to the ill-posedness of the Cauchy problem. Joly, Métivier and Rauch distinguish "direct focusing" of principal phases and "hidden focusing" resulting from several interactions. They define a *coherence* criterium so as to separate clearly eikonal and non-eikonal phases.

Definition 2.1. *A real vector space $\Phi \subset \mathcal{C}^\infty(\Omega)$ is L-coherent when for all $\varphi \in \Phi \backslash \{0\}$, one of the following conditions holds:*
(i) $\det L(t, x, \mathrm{d}\varphi(t,x)) = 0$ and $\mathrm{d}\varphi(t,x) \neq 0$ for every $(t,x) \in \Omega$,
(ii) $\det L(t, x, \mathrm{d}\varphi(t,x))$ does not vanish on Ω.

The typical example of L-coherence is for linear phases and constant coefficient operator L. Coherence is not easy to check on a given set of phases, but it allows to prove (locally, on truncated cones $\Omega_T = \{(t,x) \mid 0 \leq t \leq T, \ |x| + t/\delta \leq \rho\}$, for small $\delta, \rho > 0$), for a symmetric hyperbolic system with constant multiplicity, the following result.

Theorem 2.1. *Consider initial phases $\vec{\varphi}^{\ 0} = (\varphi_1^0, \ldots, \varphi_q^0)$ such that $\mathrm{d}\varphi_k^0$ does not vanish on Ω_0. Consider eikonal phases $\vec{\varphi} = (\varphi_1, \ldots, \varphi_q)$ generated by these initial phases ($\forall k, \ \varphi_k(0,x) = \varphi_k^0(x)$), and assume that their linear span Φ is L-coherent, and contains a timelike phase φ_0: $\varphi_0(0,x) = 0$, and $\partial_t \varphi_0$ does not vanish. Then, for any initial data*

$$(10) \qquad\qquad h^\varepsilon(x) = \varepsilon^m \mathcal{H}^\varepsilon(x, \vec{\varphi}^{\ 0}/\varepsilon),$$

9

with $(\mathcal{H}^\varepsilon)_{0<\varepsilon\leq 1}$ a bounded family in the Sobolev space $H^s(\overline{\Omega_0} \times \mathbb{T}^q)$ for some $s > 1 + (d+q)/2$, we have
(i) There is $T > 0$ such that for all $\varepsilon \in]0,1]$, the Cauchy problem (2), (10) has a unique solution $u^\varepsilon \in \mathcal{C}^1(\Omega_T)$.
(ii) There is a profile $\mathcal{U}(t, x, \tau, \theta) \in \mathcal{C}^1(\Omega_T \times \mathbb{R} \times \mathbb{T}^q)$, almost periodic in (τ, θ) (and determined by the profile equations (8a), (8b)) such that

$$u^\varepsilon(t,x) - \varepsilon^m \mathcal{U}(t, x, \varphi_0/\varepsilon, \vec{\varphi}/\varepsilon) = o(\varepsilon^m) \ in \ L^\infty.$$

The new idea in their proof consists in looking for exact solutions of the form
$$u^\varepsilon(t,x) = \varepsilon^m \mathcal{U}^\varepsilon(t, x, \vec{\varphi}/\varepsilon).$$
It is then sufficient (for u^ε to be a solution to (2)) that the profile $\mathcal{U}^\varepsilon$ be solution to the *singular system*

$$L_1(\partial_{t,x})\,\mathcal{U}^\varepsilon + \frac{1}{\varepsilon}\left(\sum_{k=1}^q L_1(\mathrm{d}\varphi_k)\partial_{\theta_k} + L_0\right)\mathcal{U}^\varepsilon = F^\varepsilon(t, x, \mathcal{U}^\varepsilon),$$

and L-coherence precisely allows energy estimates in H^s.

The result of Joly, Métivier and Rauch on singular systems generalizes the ones of Klainerman and Majda [KM81], [KM82] and Schochet [Sch94a]. It is worth noting that the analysis of such singular systems appears in many multi-scale problems. In particular, for rapidly rotating fluids in oceanology, meteorology, and Magneto-Hydro-Dynamics; see [BMN96], [Gre97b], [Gal98], [MS01]; several applications are described by Klein [Kle00]. This method also provides an efficient tool for numerics: see the works of Colin and Nkonga about propagation of wavetrains [CN04b] and pulses (see section 3) [CN05] in optical media, and Colin and Torri [CT05] about pulse propagation over diffractive scales (see section 6.2).

A work close to the method exposed here may be found in the paper [CCar] by Colin and Colin, studying rigorously and numerically Raman scattering for a semi-classical quasilinear Zhakarov system from plasma physics in the weakly nonlinear geometric optics regime, with three-wave resonances. There, the system lacks hyperbolicity, which is compensated by dispersion.

It is in fact possible to achieve infinitely accurate asymptotics for these multidimensional nonlinear interacting waves with nonlinear phases, by adding to the coherence assumption on phases a generically satisfied "not too small divisors" hypothesis (or Diophantine hypothesis on wave vectors, in the case of linear phases), see [JMR93a].

3   Other kinds of profiles

As we have seen, the natural dependence of the profiles $U(t, x, \theta)$ on the fast variable $\theta \in \mathbb{R}^q$ is periodic, quasi- or almost periodic. This allows to define an oscillating spectrum [Jol83], [JMR94] which is localized on the frequencies given by the Fourier transform of $U(t, x, \cdot)$. Now, WKB aymptotics with profiles having other behaviors w.r.t. θ may be relevant, depending on the context. The formal computations are in general very similar to the ones of usual geometric optics, but at least, the functional tools (such as average operators) must be re-defined, and interactions may take a different form.

In order to model the optical Raman scattering, for which light is emitted in a continuum of directions, Lannes [Lan98] introduces profiles with continuous spectra, and gives the analogue of usual rigorous weakly nonlinear geometric optics in this context, with a precise analysis of resonances.

In fact, this formalism also includes (see Barrailh and Lannes [BL02]) the one of "ultrashort pulses" (from laser physics) considered by Alterman and Rauch [AR02], where profiles have a compact support in θ (see also sections 4 and 6 about focusing and diffraction of pulses).

Sometimes, profiles may also have different limits as the variable $\theta \in \mathbb{R}$ goes to $\pm\infty$, in order to match boundary conditions (see Section 5.2), or to describe transitions like the ones of [Dum04], between light and shadow (see Section 6); this is also the case of the solitary waves in [Guè95a].

4   Caustics

In space dimension greater or equal to 2, singularities appear, even in the case of linear geometric optics, in the resolution of the eikonal equation: when rays have an envelope (called a caustic; the example of the cusp $\{(t, x_1) \in [0, \infty[\times \mathbb{R} \mid t^{2/3} = x_1^{2/3} + 1\}$ from the wave equation in space dimension 2 and $\varphi(0, x) = x_2 + x_1^2$ is shown on Figure 1), the second derivative of the phase φ becomes singular. In this case, the amplitude is also singular [Lud66].

In order to solve the eikonal equation globally in time, one considers the Lagrangian manifold foliated by the Hamiltonian flow associated with $L(t, x, \tau, \xi)$, starting from points $(0, x, 0, \mathrm{d}\varphi(0, x))$. Caustic crossing induces a phase shift determined by Maslov's index, and generates new phases (see Duistermaat [Dui74]; one phase before the cusp of Figure 1 corresponds to three phases beyond the cusp).

Following these ideas, Joly, Métivier and Rauch have studied the caustic crossing for semilinear geometric optics. In [JMR95b], they show that, for superlinear nonlinearities and focusing at a point, oscillations may lead to explosion, whereas in the case of dissipative equations (for which exact solutions are globally defined), they may be absorbed (*i.e.* only a non-oscillating term remains after reflection on the caustic). In [JMR00a], for dissipative equations, they extend this result to general caustics. They exhibit a critical exponent p_c, defined by the geometry of the caustic so that, if the nonlinearity is stronger than $|u|^{p_c}$ at infinity, absorption occurs, and else, oscillations persist. In [JMR96], for uniformly Lipschitzean nonlinearities (for which, again, exact solutions are globally defined), they show that oscillations go through the caustic.
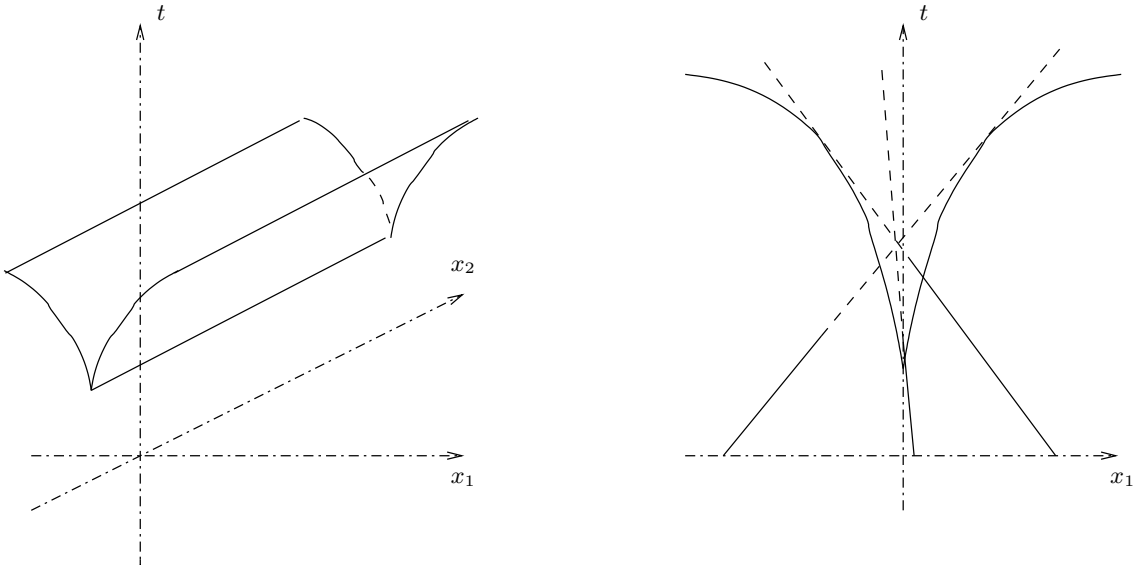

Figure 1: The cusp. Through each (t, x) point pass only one ray before the caustic, and three rays beyond the caustic.

More precisely, to a single-phase expansion before the caustic (case considered here just for notational simplicity; several initial phases are allowed),

$$u^\varepsilon(t, x) = \varepsilon^m \mathcal{U}(t, x, \varphi/\varepsilon) + o(1),$$

12

corresponds a multiphase expansion beyond the caustic,

$$u^\varepsilon(t,x) = \varepsilon^m \underline{u}(t,x) + \varepsilon^m \sum_j \mathcal{U}(t,x,\varphi_j/\varepsilon) + o(1),$$

with phases φ_j defined as in the linear case, matching φ on the caustic. In particular, this provides examples of asymptotics with non-coherent phases. This approximation is in L^2 in [JMR95c] and [JMR96], and in all L^p's, $p < \infty$, in [JMR00a]. It is based on the oscillating integral representation

$$u^\varepsilon(t,x) = \varepsilon^m \underline{u}(t,x) + \varepsilon^m \int e^{i\phi(t,x,y,\xi)/\varepsilon} \mathcal{A}(t,x,\varphi/\varepsilon) \, \mathrm{d}y \, \mathrm{d}\xi + o(1)$$

$$=: \varepsilon^m \underline{u}(t,x) + \varepsilon^m I^\varepsilon(\mathcal{A}) + o(1).$$

Outside the caustic, expanding a smooth $\mathcal{A}(t,x,\theta)$ in Fourier series in θ and performing the usual stationary phase asymptotics (the assumption that the phases φ_j are non-resonant is needed) gives

$$I^\varepsilon(\mathcal{A}) \sim \sum_j J_j^\varepsilon(\mathcal{A}),$$

where the sum corresponds to the several pre-images of critical points of the phase, and $J_j^\varepsilon(\mathcal{A})(t,x)$ is the evaluation at $(t,x,\varphi_j/\varepsilon)$ of some profile $\mathcal{U}_j(t,x,\theta)$ obtained from $\mathcal{A}$ *via* a Hilbert transform.

Here, the amplitude $\mathcal{A}$ belongs to L^p, so that the profiles $\mathcal{U}_j$ are in fact better defined as "weak profiles" (in particular, $\mathcal{U}_j(t,x,\varphi_j/\varepsilon)$ has an asymptotic sense only). They are weak limits of u^ε, using $J_j^\varepsilon(\mathcal{B})$ as test function, for smooth $\mathcal{B}(t,x,\theta)$,

$$\forall \mathcal{B}(t,x,\theta), \quad \int \overline{u^\varepsilon} J_j^\varepsilon(\mathcal{B}) \mathrm{d}x \xrightarrow[\varepsilon \to 0]{} \int \overline{\mathcal{U}_j} \mathcal{B} \mathrm{d}x \mathrm{d}\theta.$$

Profile equations are then obtained as weak limits of the original equation, using a nice "no propagated oscillations" lemma ([JMR00a], section 5.2) for nonlinearities. Strong convergence then follows from energy estimates and L^p estimates on oscillatory integrals.

These results are refined (in L^∞ instead of L^p) and extended to conservative systems by Carles, for simpler geometries. Essentially, Carles considers focusing on a point for the wave equation,

(11) $$\Box u^\varepsilon + a|\partial_t u^\varepsilon|^{p-1} \partial_t u^\varepsilon = 0, \quad p > 1, \ a \in \mathbb{C},$$

13

($a > 0$, $a < 0$, $a \in i\mathbb{R}$ corresponds to the dissipative, accretive and conservative case, respectively) with radially symmetric initial data, or the semiclassical nonlinear Schrödinger equation (NLS),

$$(12) \qquad i\varepsilon\partial_t u^\varepsilon + \frac{1}{2}\varepsilon^2 \Delta u^\varepsilon = \varepsilon^\alpha |u^\varepsilon|^{2\sigma} u^\varepsilon, \quad \alpha \geq 1, \ \sigma > 0,$$

with initial data of the form

$$u^\varepsilon_{|t=0} = f(x)e^{-i|x|^2/2\varepsilon}.$$

Hunter and Keller [HK87] give a formal classification of the qualitative properties of weakly nonlinear geometric optics for (11), separating linear and nonlinear propagation, and linear and nonlinear effects of the caustic. Carles has rigorously explained this classification. In [Car98], for $1 < p < 2$, he improves the results of [JMR00a] ([JMR95b] corresponds to $a > 0$, $p > 2$) with an L^∞ description of radial waves in $\mathbb{R}^3$. He shows that the profiles are really singular, and that new amplitudes (of size ε^{1-p}) must be added to the one in [HK87] near the caustic.

In a series of papers [CR02], [CR04a], [CR04b], Carles and Rauch study the focusing of spherical pulse solutions to (11) in space dimension 3, for various powers p and sizes (ε^m) of initial data. In particular, they show how pulses get wider after reflection on the caustic; they also treat the analogue of [JMR95b] ($a \in \mathbb{R}$) and find the same results (dissipation or growth) as for wavetrains. Carles and Lannes generalize these results [CL03a] to dispersive semilinear systems such as Klein-Gordon and Hartree's equations. Again for (11), they obtain [CL03b] the focusing (at $t = 1$) of a pulse with "random phase shift" $\ln((1-t)/\varepsilon)$ –in the conservative case, for nonlinear propagation and a nonlinear caustic.

Concerning NLS (12), Carles investigates all the behaviors with different powers $\alpha \geq 1$ and $\sigma > 0$. In [Car01], he shows the possibility of "random phase shift". In [Car00b] ($\alpha = d\sigma > 1$, in space dimension d), he notices that nonlinear effects take place near the focal point only; thus, in [CFKG03], [CKar], he answers with his co-authors the converse question "for which kind of initial data can we get nonlinear effects at first order?". He also considers [Car00a] focusing on a line instead of a point.

Other extensions of [Car00b] concern the description of Bose-Einstein condensates, *via* the addition to (12) of a polynomial potential [Car03], [CM04], [CN04a], or Hartree-type nonlinearities $((1/|x|) \star |u|^2)u$ [CMSar] for the Schrödinger-Poisson model.

The method of Carles consists in a precise control of the solution outside the caustic *via* WKB expansions and Gagliardo-Nirenberg estimates, then rescaling the problem around the focal point so as to use nonlinear scattering arguments (with short-range or long-range interpretation, according to the power of the nonlinearity).

We may mention that caustics are subject to many numerical works, such as the ones of Benamou *et al.*, [BS00], [BH02], [Ben03], [BLSS03], [BLSS04], based on a Eulerian –or ray tracing– approach. See also the review of Gosse [Gos05] –about (NLS), mostly with kinetic methods– and references therein.

5 Boundary problems

5.1 Reflection on a boundary

The formal study of reflections of weakly nonlinear waves is done by Artola and Majda in [MA88]. In the case of transverse reflection, rigorous results are due to Chikhi [Chi91] for equations with two speeds of propagation, and to Williams [Wil93] for general resonant interactions.

The first order tangential reflection corresponds to diffractive points in the cotangent bundle of the boundary, and leads to the formation of a shadow zone (in fact, a caustic forms in the obstacle, so that phases become singular at the tangency point). The linear case is treated by Friedlander [Fri76] and Melrose [Mel75a], [Mel75b]. Weakly nonlinear geometric optics is justified, at first order, with nonlinear incident and reflected phases, by Cheverry [Che96b] for globally Lipschitz nonlinearities (constructing smooth profiles, solutions to singular ODE's along the broken flow of the rays), and by Dumas [Dum02] for a dissipative nonlinear Klein-Gordon equation (using weak profiles as described in section 4).

Higher order tangency is studied by Williams [Wil93], [Wil00], in the case of a simpler geometry (flat boundary, constant coefficient operators and linear phases). He obtains L^∞ asymptotics at all orders, incorporating in the description boundary layer profiles (see section 5.2) involving a third scale $\sqrt{\varepsilon}$: profile equations are then of Schrödinger type (see section 6).

5.2 Boundary layers

Boundary layers are a usual feature in the study on partial differential equations on domains with boundaries: they stem from large variations of the solution in a small region near the boundary. They often appear in vanishing viscosity models, since the boundary conditions are not the same for the viscous (parabolic) and inviscid (hyperbolic) equations.

Guès [Guè95b] describes such boundary layers for viscous perturbations $(-\varepsilon^2 \mathcal{E}$, with $\mathcal{E}(t, x, \partial)$ a second order elliptic operator) of semilinear mixed problems (associated with a linear hyperbolic operator $L(t, x, \partial)$). He constructs exact solutions with infinitely accurate WKB asymptotics of the form

$$u^\varepsilon(t, x) \sim \sum_{n \geq 0} \sqrt{\varepsilon}^{\; n} \, \mathcal{U}_n\left(t, x, \varphi/\sqrt{\varepsilon}, \varphi/\varepsilon\right),$$

$$\mathcal{U}_n\left(t, x, \theta, z\right) = \underline{\mathcal{U}}_n(t, x) + \mathcal{V}_n(t, x, \theta) + \mathcal{W}_n(t, x, z) + \chi_n(t, x, \theta, z),$$

where the phase φ is transverse to the boundary, and functions of $\theta, z \geq 0$ decay rapidly at $+\infty$ (so that $\underline{\mathcal{U}}_n = \lim\limits_{\theta, z \to +\infty} \mathcal{U}_n$). Profile equations are analogue to the ones of geometric optics (but of parabolic type), with polarization conditions. When the boundary is non-characteristic for the hyperbolic operator L, the dependence on $\theta = \varphi/\sqrt{\varepsilon}$ is not necesseray, and the boundary layer has size ε; in the characteristic case, the boundary layer is wider, of size $\sqrt{\varepsilon}$. Grenier [Gre97a] considers the quasilinear case with characteristic boundary, and Grenier and Guès [GG98], the non-characteristic quasilinear case. Extensions may be found in the work of Sueur for semilinear systems [Sueara], [Suearc], [Sue05b], or quasilinear systems [Suearb], [Sue05a].

When oscillations are present in the interior domain, they may interact with boundary layers. This occurs for example in the study of glancing oscillations by Williams [Wil00]. It also occurs for rotating fluid models (see the books of Greenspan [Gre80] and Pedlovsky [Ped79] about oceanography and meteorology, and Dormy [Dor97] about magneto-hydro-dynamics), which in the incompressible case typically have the form

$$\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \frac{1}{\varepsilon} R u^\varepsilon - \varepsilon \Delta u^\varepsilon = -\nabla p^\varepsilon,$$

where p^ε is the pressure, u^ε is the divergence free velocity field, and the operator R is skew-symmetric (for example, $Ru = e \times u$ with a fixed vector e for Coriolis effect). Initial data are well-prepared when they correspond

to admissible data for the limit problem. Otherwise, fast time oscillations appear in the interior domain, and are taken into account in the profiles through a dependence on the variable t/ε. For such studies in the well-prepared case, see [GM97]; in the ill-prepared case, see [Mas00], [CDGG02], [BDGV04], [GV03b]; Gérard-Varet emphasizes the link with geometric optics in [GV03a] and [GV05].

This technique is also used for micromagnetism. Carbou, Fabrie and Guès [CFG02] describe the magnetization of a ferromagnetic medium, given by Landau-Lifshitz equation [LL69], as the exchange coefficient goes to zero; Sanchez [San02] studies the diffraction of an electromagnetic wave by a conducting domain, as the conductivity goes to infinity.

5.3 Shocks

Shock waves entering the framework of weakly nonlinear geometric optics are weak shocks, since their amplitude goes to zero with ε. Such an oscillatory shock is a perturbation of a non-oscillating shock $(\underline{u}^\pm, \underline{\psi})$, and is given (in space dimension d, with $x = (x', x_d)$) by a $\mathcal{C}^1$ hypersurface,

$$x_d = \psi^\varepsilon(t, x') \sim \underline{\psi}(t, x) + \varepsilon^m \sum_{n \geq 0} \varepsilon^n \psi_n(t, x', \vec{\varphi}^{\,0}(t, x')/\varepsilon),$$

and on each side $x_d \gtrless \psi^\varepsilon(t, x')$, a $\mathcal{C}^1$ solution $u_\pm^\varepsilon$ to a system of conservation laws,

$$u_\pm^\varepsilon(t, x) \sim \underline{u}^\pm(t, x) + \varepsilon^m \sum_{n \geq 0} \varepsilon^n \mathcal{U}_n^\pm(t, x, \vec{\varphi}^{\,\pm}(t, x)/\varepsilon),$$

together with the usual Rankine-Hugoniot jump conditions. Of course, the unknowns are $u_\pm^\varepsilon$ and ψ^ε, so that this is a free-boundary problem.

A formal description is due to Artola and Majda [MA88]. Rigorous results in space dimension one are given by Corli [Cor95] in a multiphase context, with curved phases, using the same "Picard iterates" method as in [JMR93b] (and the same transversality assumptions on phases). Williams [Wil99] obtains, in the spirit of [JMR93a], multidimensional infinitely accurate asymptotics with smooth profiles, around a planar shock $(\underline{u}^\pm = cst, \underline{\psi} = \sigma t)$, which is uniformly stable in the sense of Kreiss. This is done for linear phases (with a Diophantine assumption on their gradients), whose restrictions to the shock surface are entire multiples of a single phase φ^0.

We also mention the work of Guès, Métivier, Williams and Zumbrun on the existence and stability of multidimensional viscous shocks [GMWZ04a]. These (non-oscillating) shocks are not weak ones (their size does not depend on a small parameter), and the system of conservation laws is perturbed by a viscous term $-\varepsilon \Delta u^\varepsilon$, so that the solution u^ε is smooth. The question is "are the shock solutions of the inviscid system approximated by solutions of the viscous system?". The answer is "yes" [GMWZ04b], under the assumption of the existence of a spectrally stable wave profile $U(x', (x_d - \psi)/\varepsilon)$ solution to the viscous system such that $U(x', z) \underset{z \to \pm\infty}{\longrightarrow} u^\pm$. These authors also study [GMWZ05b], [GMWZ05a] the long time stability of planar shocks, using WKB asymptotics for the viscous problem.

6 Three (and more) scale expansions: diffractive optics

So far, we have essentially presented problems where only two scales appear (at least at first sight). But more scales may be present (remember how a third scale appears with boundary layers in section 5.2). In this section, we shall see that a third scale leads to supplement the transport equations (with finite propagation speed) for the profiles with a Schrödinger-type equation (with infinite propagation speed!) taking diffraction into account; this is the usual "paraxial approximation" of laser physics in Kerr media.

6.1 Some three-scale problems

In Donnat's thesis [Don94], a model of "light cigars" is described for laser propagation. These are modulated oscillating wavetrains with frequency $1/\varepsilon$, with anisotropic "support", *i.e.* of typical size 1 along a direction, and $\sqrt{\varepsilon}$ along another. Such profiles are used by Boucherès, Colin, Nkonga, Texier and Bourgeade [BCN⁺04] to describe the interaction of a laser with a gas, giving an approximation of Maxwell-Bloch equations (5) by a nonlinear Schrödinger-Bloch system.

"Singular rays" of weakly nonlinear geometric optics may also be analysed with three-scale asymptotics. They are hypersurfaces (defined as $\psi(t, x) = 0$ for an unknown function ψ) constituted by rays meeting an obstacle tangentially or at a corner. They model transitions between light and shadow (or

sound and silence, in the case of acoustic waves). Hunter [Hun88] introduces formal WKB expansions,

$$(13) \qquad u^\varepsilon(t,x) \sim \varepsilon^m \sum_{n \geq 0} \sqrt{\varepsilon}^{\ n} U_n(t,x,\psi/\sqrt{\varepsilon},\vec{\varphi}/\varepsilon),$$

with profiles periodic w.r.t. $\theta = \varphi/\varepsilon$, and with limits as $\eta = \psi/\sqrt{\varepsilon} \to \pm\infty$ (such a u^ε may be understood as an smooth approximation of a contact discontinuity, *i.e.* a characteristic shock along $\psi = 0$). With an approach close to the one of [DJMR96], Dumas [Dum04] validates this asymptotics rigorously, introducing new coherence assumptions on the phases $(\tilde{\psi},\vec{\varphi})$, where $\tilde{\psi} = (t,\psi)$ –it turns out that an intermediate phase $\psi_0 = t$ is in general necessary for the profile equations to be well-posed. Profile equations are then transport equations in variables (t,x), and Schrödinger equations with time $\tau = t/\sqrt{\varepsilon}$ and space variables x.

Three-scale asymptotics of the form (13) also describe oscillations according to perturbed phases (in a more restricted sense than in [Gre98]). For initial data such as

$$u^\varepsilon_{|t=0} = \varepsilon^m \mathcal{U}^0(x,\varphi^0_\varepsilon/\varepsilon) \text{ with } \varphi^0_\varepsilon = \varphi^0 + \sqrt{\varepsilon}\psi^0,$$

the solution to the Cauchy problem cannot in general be approximated in L^∞ by a two-scale geometric optics description, but profiles as above, satisfying a NLS equation on a torus, achieve such an approximation –see Dumas [Dum03b].

Now, we turn to the problem of long-time validity for geometric optics, in which the third scale is not present initially, but naturally comes up. This is the context where first rigorous three-scale asymptotics were proven in nonlinear optics.

6.2   Long-time behavior

A priori, asymptotics from section 2 are valid only on some fixed time interval $[0,t_0]$. In fact, when the profiles are globally defined, uniformly bounded in space-time, and have H^s norms with polynomial growth in time, Lannes and Rauch prove [LR01] that weakly nonlinear geometric optics is valid up to times $t \sim C \ln(1/\varepsilon)$ –in the linear case, see the study of propagation on this so-called Ehrenfest time for semi-classical Schrödinger equations by De Bièvre and Robert [DBR03].

Diffractive optics. However, in order to describe oscillatory waves with frequency $1/\varepsilon$ on larger propagation scales, one must adapt the geometric optics approach (even in the linear case). First, the cumulated effects of nonlinearities over longer times leads to consider smaller amplitudes. The first natural long-time scale is then the *diffractive time scale* $1/\varepsilon$, and formula (4) is replaced with

$$(14) \qquad\qquad m = \frac{2}{J-1}.$$

Next, new variables are introduced so as to capture evolution up to $t \sim 1/\varepsilon$, $X = \varepsilon x$ and the slow time $T = \varepsilon t$.

Note that a solution to the eikonal equation is in general not smooth globally in time, unless the operator L has constant coefficients and the phase is linear, of the form $\varphi(t,x) = \beta \cdot x - \omega t$.

Formal three-scale asymptotics for solutions u^ε to (2) are given by Donnat [Don94],

$$(15) \qquad u^\varepsilon(t,x) \sim \varepsilon^m \sum_{n \geq 0} \varepsilon^n U_n(T,X,t,x,\theta)_{|(T,X,\theta)=(\varepsilon t, \varepsilon x, (\beta \cdot x - \omega t)/\varepsilon)}.$$

The natural dependence of the profiles U_n (periodic in θ) in the variable X (and x) is of Sobolev type, and continuous (thus bounded) in $T \in [0, T_0]$, whereas for consistency of the Ansatz, *i.e.* $|\varepsilon U_{n+1}| \ll |U_n|$, *sublinear growth* in $t = T/\varepsilon$ is required,

$$\forall n \geq 1, \quad \frac{1}{t} \|U_n\|_\infty \underset{t \to \infty}{\longrightarrow} 0.$$

Donnat, Joly, Métivier and Rauch [DJMR96] give a rigorous justification of this monophase approximation, for semilinear hyperbolic (non-dispersive) systems, and profiles with no non-oscillating part (assuming that the Taylor expansion of the nonlinearities contains odd parts only). The Schrödinger profile equation (see (18)) is interpreted as a diffractive correction to geometric optics on long times, by analogy with Fresnel's diffraction.

In order to get faster numerics for nonlinear optics, Colin, Gallice and Laurioux (Barrailh) [CGL05] introduce intermediate models between geometric and diffractive optics, keeping a dependence of the eigenvalue λ on ε, instead of Taylor expanding up to the order 2, so that the group velocity becomes a pseudo-differential operator $v(k + \varepsilon \partial_x)$.

Rectification. For general systems, even if the initial data is purely oscillatory, a non-oscillating mode (or mean field) will be created. This is optical *rectification*. In [JMR98], Joly, Métivier and Rauch allow such a phenomenon, still for non-dispersive systems. The case of semilinear dispersive systems is treated by Lannes [Lan98]. We now illustrate the difficulty caused by interaction of oscillating and non-oscillating modes, and simply look at the obtention of the first profile equations.

In the dispersive case, $\pi_0 \neq \mathrm{Id}$ (the origin belongs to the characteristic variety $\mathcal{C}$ of L) and the set of nonzero eikonal frequencies,

$$\mathcal{E} = \{\alpha \in \mathbb{Z}^\star \mid \alpha(-\omega, k) \in \mathcal{C}\},$$

is finite, in general. We assume that the only singular point of C in $\{\alpha(-\omega, k) \mid \alpha \in \mathcal{E}\}$ is the origin. Splitting profiles $U_n = \sum_{\alpha \in \mathbb{Z}} U_n^\alpha(T, X, t, x)\ e^{i\alpha\theta}$ into average U_n^0 and oscillations $U_n^\star = U_n - U_n^0$, one gets from the first powers of ε in the formal expansion of $Lu^\varepsilon - F(u^\varepsilon)$,

(16a) $\Pi\ U_0^\star = U_0^\star, \qquad \pi_0\ U_0^0 = U_0^0,$

(16b) $\pi_0 L_1(\partial_{t,x})\pi_0\ U_0^0 = 0,$

(16c) $\pi_0 L_1(\partial_{T,X})\pi_0\ U_0^0 - \pi_0 L_1(\partial_{t,x})L_0^{-1}L_1(\partial_{t,x})\pi_0\ U_0^0$
$$+ \pi_0 F_0(U_0^\star, U_0^0) = -(\partial_t + \vec{v}_\alpha \cdot \partial_x)\ U_1^\alpha,$$

and $\forall \alpha \in \mathbb{Z}\backslash\{0\}$,

(17a) $(\partial_t + \vec{v}_\alpha \cdot \partial_x)\ U_0^\alpha = 0,$

(17b) $(\partial_T + \vec{v}_\alpha \cdot \partial_X)\ U_0^\alpha - \pi_\alpha L_1(\partial_{t,x})L(i\alpha\beta)^{-1}L_1(\partial_{t,x})\pi_\alpha\ U_0^\star$
$$+ \pi_\alpha F_\alpha(U_0^\star, U_0^0) = -(\partial_t + \vec{v}_\alpha \cdot \partial_x)\ U_1^\alpha.$$

As for the transport operator at the group velocity, an "algebraic lemma" [Tex04] shows that the second order operator $\pi_\alpha L_1(\partial_{t,x})L(i\alpha\beta)^{-1}L_1(\partial_{t,x})\pi_\alpha$ is in fact $\frac{i}{2}\partial_\xi^2\lambda(\alpha\beta)\cdot(\partial_x, \partial_x)$. On the other hand, the operator $\pi_0 L_1(\partial_{t,x})\pi_0$, corresponding to low frequencies, and called "long-wave operator", is symmetric hyperbolic, but is not a transport operator, since the origin is a singular point of the characteristic variety $\mathcal{C}$ of L. We simply know [Lan98] that its characteristic variety $\mathcal{C}_{LW}$ is the tangent cone to $\mathcal{C}$ at the origin. This cone may contain hyperplanes $\{(\tau, \xi) \in \mathbb{R}^{1+d} \mid \tau + \vec{v}_\alpha \cdot \xi = 0\}$; this is precisely the rectification criterium.

The difficulty now lies in the compatibility of the profile equations. Lannes [Lan98] defines nice average operators, which are the analytic analogue of the

algebraic projector Π. They identify and separate the various propagation modes at scale (T, X) and give necessary and sufficient conditions for the profile equations to be solvable, with a t-sublinear corrector U_1. Split the set of eikonal frequencies $\mathcal{E}$ into the resonant set

$$\mathcal{E}_R = \{\alpha \in \mathcal{E} \mid \{\tau + \vec{v}_\alpha \cdot \xi = 0\} \subset \mathcal{C}_{LW}\},$$

–denoting $U_0^{0,\alpha}$ the corresponding modes of U_0^0, and $U_0^{0,\alpha'}$, $\alpha' \in \mathcal{E}'$, the others– and the non-resonant set, $\mathcal{E}_{NR} = \mathcal{E} \backslash \mathcal{E}_R$. Now, the abovementioned conditions consist in replacing (16c), (17b), for a resonant mode $\alpha \in \mathcal{E}_R$, with

(18a)
$$\pi_0 L_1(\partial_{T,X})\pi_0 \ U_0^{0,\alpha} - \pi_0 L_1(\partial_{t,x})L_0^{-1}L_1(\partial_{t,x})\pi_0 \ U_0^{0,\alpha} + \pi_0 F_0(U_0^\alpha, U_0^{0,\alpha}) = 0,$$
(18b)
$$(\partial_T + \vec{v}_\alpha \cdot \partial_X) \ U_0^\alpha - \frac{i}{2}\partial_\xi^2(\alpha\beta) \cdot (\partial_x, \partial_x) \ U_0^\alpha + \pi_\alpha F_\alpha(U_0^\alpha, U_0^{0,\alpha}) = 0,$$

whereas equations for non-resonant modes ($\alpha \in \mathcal{E}_{NR}, \alpha' \in \mathcal{E}'$) are decoupled from the others,

(19a)
$$\pi_0 L_1(\partial_{T,X})\pi_0 \ U_0^{0,\alpha'} - \pi_0 L_1(\partial_{t,x})L_0^{-1}L_1(\partial_{t,x})\pi_0 U_0^{0,\alpha'} + \pi_0 F_0(0, U_0^{0,\alpha'}) = 0,$$
(19b)
$$(\partial_T + \vec{v}_\alpha \cdot \partial_X) \ U_0^\alpha - \frac{i}{2}\partial_\xi^2(\alpha\beta) \cdot (\partial_x, \partial_x)U_0^\alpha + \pi_\alpha F_\alpha(U_0^\alpha, 0) = 0.$$

The corrector ΠU_1 absorbs the difference between these equations and (16c), (17b),

$$\pi_0 L_1(\partial_{t,x})\pi_0 \ U_1^0 = \pi_0 \left[F_0(U_0^\star, U_0^0) - \sum_{\alpha \in \mathcal{E}} F_0(U_0^\alpha, U_0^{0,\alpha}) \right],$$
$$(\partial_t + \vec{v}_\alpha \cdot \partial_x) \ U_1^\alpha = \pi_\alpha[F_\alpha(U_0^\star, U_0^0) - F_\alpha(U_0^\alpha, U_0^{0,\alpha})], \quad \forall \alpha \in \mathcal{E}.$$

This mode decoupling produces a solvable system of profile equations (polarization (16a), linear hyperbolic evolution (16b), (17a), slow nonlinear evolution (18)), (19)), and clearly explains the rectification effect: even if U_0^0 vanishes initially, non-oscillating modes may be created by nonlinear interactions in (18a). The asymptotics is then only at first order: the profile U_0

is constructed, together with correctors U_1, U_2, whose secular growth (see [Lan03b]) prevents from higher order asymptotics.

The analogue study for hyperbolic systems with variable coefficients is done by Dumas [Dum03b], [Dum04], for WKB expansions with several non-linear phases: a rescaling in (13) leads to a "weakly nonplanar" version of long-time diffraction,

$$u^\varepsilon(t, x) \sim \varepsilon^m \sum_{n \geq 0} \varepsilon^n U_n(\varepsilon t, \varepsilon x, \vec{\psi}(\varepsilon t, \varepsilon x)/\varepsilon, \vec{\varphi}(\varepsilon t, \varepsilon x)/\varepsilon^2).$$

The validation of such asymptotics necessitates coherence assumptions on the phases $\vec{\varphi}$, $\vec{\psi}$ relatively to the operator L as well as to the tangent operators $\pi_\alpha L \pi_\alpha$.

Self-focusing. Solutions to a (focusing) nonlinear Schrödinger equation may blow up in finite time, and this is usually interpreted in laser physics as the self-focusing of the laser beam [SS99]. Now, if the solution to the original Mawxell equations describing the beam is globally defined, what does this singularity mean? In [Dum03a], Dumas shows that the diffractive profile blow-up corresponds to focusing for a perturbed weakly nonlinear geometric optics model, and Schrödinger approximation is valid at least up to a time t_ε of the order of a negative power of $\ln(1/\varepsilon)$ before blow-up. This shows that, even if each solution u^ε is defined globally in time, it undergoes, between $t = 0$ and $t = t_\varepsilon$, an amplification by a positive power of $\ln(1/\varepsilon)$.

Pulses and continuous spectra. The propagation of pulses over diffractive times is described by Alterman and Rauch in the linear [AR00] and nonlinear case [AR03]. The profiles from (15) then have compact support in θ. An important difference with diffractive optics for oscillatory wavetrains resides in the profile equations

$$\partial_T \partial_z U - \frac{1}{2} \partial_\xi \lambda(\partial_z)(\partial_x^2 + \partial_y^2)U = F(U).$$

The operator ∂_z^{-1} is not defined on the space of profiles considered, which can be seen, by Fourier transform, as a small divisors problem. Alterman and Rauch solve this difficulty using "infrared cut-offs": they define approximate profiles by truncating the low frequencies in the equation, and then show the convergence of the approximate solution based on these approximate

profiles. See also [SU03]: even if Schneider and Uecker do not compare their results with the ones from the WKB method, they address the problem of existence and stability of diffractive pulse solutions to nonlinear optics Maxwell's equations, using a center manifold reduction (so that they get exponential asymptotic stability, but only around some particular family of solutions).

In [BL02], Barrailh and Lannes extend this approach to profiles with continuous spectra, which model Raman scattering and lasers with large spectrum.

6.3 Transparency and larger amplitudes

Transparency. The analysis above enlights the nonlinear interactions leading to rectification. Unfortunately, computations on physical models, such as Maxwell-Bloch equations , or ferromagnetic Maxwell equations, reveal that the nonlinearities in (18a) simply vanish! This phenomenon, called *weak transparency*, is expressed as,

$$\forall U \in \mathbb{C}^N, \forall \alpha \in \mathbb{Z}, \quad \pi_\alpha \sum_{\beta \in \mathbb{Z}} F_\alpha(\pi_\beta U, \pi_{\alpha-\beta} U) = 0.$$

In order to reach nonlinear regimes, one may then increase the observation time, or the amplitude. The latter is studied (for geometric optics time $\mathcal{O}(1)$) by Joly, Métivier, Rauch in [JMR00b] for semilinear systems of Maxwell-Bloch type (including the physical ferromagnetism system in space dimension one),

$$(20) \qquad \begin{cases} L(\varepsilon\partial)u^\varepsilon + \varepsilon f(u^\varepsilon, v^\varepsilon) = 0, \\ M(\varepsilon\partial)v^\varepsilon + g(u^\varepsilon, u^\varepsilon) + \varepsilon h(u^\varepsilon, v^\varepsilon) = 0, \end{cases}$$

with L and M symmetric hyperbolic operators as in (3), and f, g, h bilinear nonlinearities. When weak transparency occurs, they look for solutions with larger amplitude than the usual ones ($\mathcal{O}(1)$ instead of ε). They show that weak transparency is necessary to construct formal WKB expansions, and give a necessary and sufficient criterium (strong transparency) for the stability of these WKB solutions. Strong transparency and the particular structure of the system in fact allow a nonlinear change of unknowns which brings back to the weakly nonlinear setting.

The same transparency property allows Jeanne [Jea02] to construct geometric optics asymptotics of large solutions to (semilinear) Yang-Mills equations from general relativity.

Using Joly-Métivier-Rauch's method, Colin [Col02] rigorously derives Davey-Stewartson (DS) systems (coupling of (NLS) and a hyperbolic or elliptic equation, modeling for example shallow-water, starting from Euler equations with free surface) from Maxwell-Bloch type systems, over diffractive times. In space dimension one, Schneider [Sch98a], [Sch98b] obtains similar results *via* normal form techniques. Colin also shows that the obstruction for such a derivation from general hyperbolic systems comes from rectification effects. Next, Colin and Lannes [CL04] perform the extension to general systems, and apply their results to the Maxwell-Landau-Lifshitz ferromagnetism model (see Landau and Lifshitz [LL69]), getting mean-field generation. This corresponds to the physics papers of Leblond on pulse propagation, deriving (NLS) [Leb01], combining with an expansion of optical susceptibilities [Leb02], or deriving (DS) in ferromagnetic media [Leb96], [Leb99].

For ill-prepared data allowing rectification, they build a (DS) approximation (valid only over times of the order of $\ln(1/\varepsilon)$). To this end, they need a long-wave correction to the original Ansatz. Furthermore, they assume (in order for the limit (DS) system to be well-posed, *i.e.* to be a Schrödinger-elliptic coupling) that long-wave–short-wave resonance (CROLOC, in French) does not occur, which means that the characteristic varieties of the long-wave operator and of the original operator are not tangent away from the origin. In order to reach $\mathcal{O}(1/\varepsilon)$ times with rectification, they consider [CL01] solutions with size $\sqrt{\varepsilon}$ (intermediate between 1 and ε), and use four-scale profiles to obtain their CROLOC. A numerical study of long-wave–short-wave resonance is due to Besse and Lannes [BL01].

Some works on waves in plasmas rely on similar techniques. The basic model is then the quasilinear Euler-Maxwell system (see Sulem and Sulem [SS99]). Colin, Ebrard, Gallice and Texier [CEGT04] study a simpler model, a Klein-Gordon-wave coupling, for which a change of unknowns leads to a semilinear system with weak transparency property. The diffractive time approximation is the Zakharov system, for electromagnetic field u, and ion population n,

$$\begin{cases} i\partial_t u + \Delta u = nu, \\ \partial_t^2 n - \Delta n = \Delta |u|^2. \end{cases}$$

For the full Euler-Maxwell system, Texier [Tex05] rigorously derives, in the

geometric optics regime, a weak form of the above system (where ε stands in front of the terms Δu and nu, thanks to "generalized WKB asymptotics", *i.e.* without eliminating the residual from profile equations.

Conservation laws with a linearly degenerate field. Cheverry, Guès and Métivier have adapted the notion of transparency above to quasilinear systems of conservation laws. They classify nonlinear regimes on times $\mathcal{O}(1)$ with asymptotics

$$u^\varepsilon(t,x) \sim u_0(t,x) + \sum_{k \geq 1} \varepsilon^{k/l} U_k(t,x,\varphi/\varepsilon).$$

The value $l = 1$ corresponds to weakly nonlinear regime; $l = 2$, to *strong oscillations*; $l \geq 3$, to *turbulent oscillations*; $l = \infty$, to *large amplitude* (*i.e.* $\mathcal{O}(1)$) oscillations (more singular solutions are the stratified solutions of Rauch and Reed [RR88], studied in the quasilinear case by Corli and Guès [CG01]). A difficulty in WKB analysis, when dealing with amplitudes larger than the weakly nonlinear ones, comes from the fact that the hierarchy of profile equations changes. In particular, a coupling appears between phase and amplitude in the eikonal equation.

In [CGM03], they show that linear degeneracy of one of the fields is a weak transparency condition that ensures existence of formal strong solutions. In [CGM04], they study existence and stability of large amplitude (polarized) waves, under a stronger transparency assumption (existence of a good symmetrizer and constant multiplicity of the linearly degenerate eigenvalue) that relates the system to Euler non-isentropic gas dynamics equations –see Métivier and Schochet [MS01] about the stability of large oscillations in time for the entropy. Earlier results on this topic were only formal ones (Serre [Ser95]), or in space dimension one: see Peng [Pen92], Heibig [Hei93], Corli and Guès [CG01], and Museux [Mus04].

Cheverry continues this work, in connection with turbulence. In [Che04], he proves that (vanishing) viscosity can compensate the lack of transparency. In [Chear], he converts the stability problem into a "cascade of phases" phenomenon, where the phase has an asymptotic expansion, whose coefficients are coupled with amplitudes.

Supercritical WKB solutions to (NLS). Coupling between phase and amplitude also occurs for large amplitude oscillating solutions to nonlinear

Schrödinger equations. Existence of such solutions before caustics is obtained by Grenier [Gre98] for a single nonlinear phase φ, with infinitely accurate WKB expansions (extending results of Gérard [Gér93]), thanks to a "perturbed phase" technique, $\varphi \sim \sum_k \varepsilon^k \varphi_k$.

7 Long waves

We finally mention the situation *a priori* opposite to highly oscillatory problems, where wavelength is large. This is the typical framework of water waves, modelled by Euler equations with free surface, (EFS); see [SS99]. However, here again come into play the coupling with a mean field, and the "long wave operator", as for diffractive optics rectification effects from previous section. This explains, for example, the formation of pairs of waves travelling in opposite directions.

Due to symmetries, the second-order differential terms vanish, in long-time asymptotic models, and the limit equations take the form of Korteweg-de Vries (KdV) equations. For one-dimensional quadratic quasilinear dispersive systems of type (2), the appropriate Ansatz is

$$u^\varepsilon(t,x) \sim \varepsilon^2 \sum_{n \geq 0} \varepsilon^n U_n(\varepsilon^2 t, t, x),$$

and profiles satisfy transport equation at scale (t,x) so that,

$$U_0(\varepsilon^2 t, t, x) = V_0(T,y)_{|T=\varepsilon^2 t, y=x\pm t}.$$

Furthermore, they are solutions to (KdV) in variables (T,y),

$$\partial_T V_0 + \frac{1}{6}\partial_y^3 V_0 + \frac{3}{4}\partial_y(V_0^2) = 0.$$

Ben Youssef and Colin [BYC00], as well as Schneider [Sch98c], derive this asymptotics for simplified general systems, getting decoupled equations for each mode. The case of (EFS) is treated by Schneider and Wayne with [SW02] or without [SW00] surface tension. The case of transverse perturbations is more singular, since it leads (for profiles $U(\varepsilon^2 t, \varepsilon x_2, t, x_1)$) to a Kadomtsev-Petviashvili-type equation (KP),

$$\partial_T V + \frac{1}{6}\partial_y^3 V + \frac{1}{2}\partial_y^{-1}\partial_Y^2 V + \frac{3}{4}\partial_y(V^2) = 0.$$

This is obtained by Gallay and Schneider for unidirectional waves, and then by Ben Youssef and Lannes [BYL02] for pairs of waves. Note that (KP) equations are singular with respect to low frequency (in y) solutions. The "infrared cut-off" technique from section 6.2 is thus useful here –and this singularity poses the problem of consistency of the approximate solution, since residuals $Lu^\varepsilon - F(u^\varepsilon)$ may not be small (see Lannes [Lan03a]).

The equations are asymptotically decoupled, but coupling appears at least when propagation takes place in a bounded domain (or for periodic solutions). Taking coupling into account also improves convergence rates. Ben Youssef and Lannes highlight such coupling effects between (KP) equations simply using "generalized WKB expansions" as described in paragraph 6.2. A well-known coupled approximation of (EFS) is the Boussinesq system. In fact, Bona, Colin and Lannes [BCL03] obtain, in 2 or 3 space dimension, a result ensuring the simultaneous validity (or non-validity) of a whole three-parameters family of such systems, including Boussinesq. For numerics on this topic, see Labbé and Paumond [LP04].

The same regime is of interest in micromagnetism, *i.e.* for Maxwell-Landau-Lifshitz system (where the medium responds to the electromagnetic field *via* magnetization). Colin, Galusinski and Kaper [CGK02] study the propagation of pairs of travelling waves in space dimension one, and derive a semilinear heat equation, whereas for two-dimensional perturbations, Sanchez [San05] obtains Khokhlov-Zabolotskaya equations.

References

[AR00]   Deborah Alterman and Jeffrey Rauch. The linear diffractive pulse equation. *Methods Appl. Anal.*, 7(2):263–274, 2000. Cathleen Morawetz: a great mathematician.

[AR02]   Deborah Alterman and Jeffrey Rauch. Nonlinear geometric optics for short pulses. *J. Differential Equations*, 178(2):437–465, 2002.

[AR03]   Deborah Alterman and Jeffrey Rauch. Diffractive nonlinear geometric optics for short pulses. *SIAM J. Math. Anal.*, 34(6):1477–1502, 2003.

[BCL03]   Jerry Bona, Thierry Colin, and David Lannes. Long wave approximations for water waves. In *Mathematical and numerical aspects of wave propagation—WAVES 2003*, pages 489–494. Springer, Berlin, 2003.

[BCN+04]   Thomas Boucherès, Thierry Colin, Boniface Nkonga, Benjamin Texier, and Antoine Bourgeade. Study of a mathematical model for stimulated Raman scattering. *Math. Models Methods Appl. Sci.*, 14(2):217–252, 2004.

[BDGV04]   Didier Bresch, Benoît Desjardins, and David Gérard-Varet. Rotating fluids in a cylinder. *Discrete Contin. Dyn. Syst.*, 11(1):47–82, 2004.

[Ben03]   Jean-David Benamou. An introduction to Eulerian geometrical optics (1992–2002). *J. Sci. Comput.*, 19(1-3):63–93, 2003. Special issue in honor of the sixtieth birthday of Stanley Osher.

[BH02]   Jean-David Benamou and Philippe Hoch. GO++: a modular Lagrangian/Eulerian software for Hamilton Jacobi equations of geometric optics type. *M2AN Math. Model. Numer. Anal.*, 36(5):883–905, 2002. Programming.

[BL01]   Christophe Besse and David Lannes. A numerical study of the long-wave short-wave resonance for 3D water waves. *Eur. J. Mech. B Fluids*, 20(5):627–650, 2001.

[BL02]   Karen Barrailh and David Lannes. A general framework for diffractive optics and its applications to lasers with large spectrums and short pulses. *SIAM J. Math. Anal.*, 34(3):636–674, 2002.

[BLSS03]   J.-D. Benamou, O. Lafitte, R. Sentis, and I. Solliec. A geometrical optics-based numerical method for high frequency electromagnetic fields computations near fold caustics. I. *J. Comput. Appl. Math.*, 156(1):93–125, 2003.

[BLSS04]   J.-D. Benamou, O. Lafitte, I. Solliec, and R. Sentis. A geometric optics method for high-frequency electromagnetic fields computations near fold caustics. II. The energy. *J. Comput. Appl. Math.*, 167(1):91–134, 2004.

[BMN96]   Anatoli V. Babin, Alex Mahalov, and Basil Nicolaenko. Resonances and regularity for Boussinesq equations. *Russian J. Math. Phys.*, 4(4):417–428, 1996.

[Boy03]   Robert W. Boyd. *Nonlinear optics*. Academic Press, 2003.

[BS00]   Jean-David Benamou and Ian Solliec. An Eulerian method for capturing caustics. *J. Comput. Phys.*, 162(1):132–163, 2000.

[BYC00]   Walid Ben Youssef and Thierry Colin. Rigorous derivation of Korteweg-de Vries-type systems from a general class of nonlinear hyperbolic systems. *M2AN Math. Model. Numer. Anal.*, 34(4):873–911, 2000.

[BYL02]   Walid Ben Youssef and David Lannes. The long wave limit for a general class of 2D quasilinear hyperbolic problems. *Comm. Partial Differential Equations*, 27(5-6):979–1020, 2002.

[Car98]   Rémi Carles. Comportements précisés près d'une caustique en optique géométrique non linéaire. *Comm. Partial Differential Equations*, 23(11-12):1929–1967, 1998.

[Car00a]   Rémi Carles. Focusing on a line for nonlinear Schrödinger equations in $\mathbf{R}^2$. *Asymptot. Anal.*, 24(3-4):255–276, 2000.

[Car00b]   Rémi Carles. Geometric optics with caustic crossing for some nonlinear Schrödinger equations. *Indiana Univ. Math. J.*, 49(2):475–551, 2000.

[Car01]   Rémi Carles. Geometric optics and long range scattering for one-dimensional nonlinear Schrödinger equations. *Comm. Math. Phys.*, 220(1):41–67, 2001.

[Car03]   Rémi Carles. Semi-classical Schrödinger equations with harmonic potential and nonlinear perturbation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(3):501–542, 2003.

[CB69]   Yvonne Choquet-Bruhat. Ondes asymptotiques et approchées pour des systèmes d'équations aux dérivées partielles non linéaires. *J. Math. Pures Appl. (9)*, 48:117–158, 1969.

[CCar]   Thierry Colin and Mathieu Colin. A numerical model for the Raman amplification for laser-plasma interaction. *J. Comp. Appl. Math.*, to appear.

[CDGG02]   Jean-Yves Chemin, Benoît Desjardins, Isabelle Gallagher, and Emmanuel Grenier. Ekman boundary layers in rotating fluids. *ESAIM Control Optim. Calc. Var.*, 8:441–466, 2002. A tribute to J. L. Lions.

[CEGT04]   Thierry Colin, Géraldine Ebrard, Gérard Gallice, and Benjamin Texier. Justification of the Zakharov model from Klein-Gordon–wave systems. *Comm. Partial Differential Equations*, 29(9-10):1365–1401, 2004.

[CFG02]   Gilles Carbou, Pierre Fabrie, and Olivier Guès. Couche limite dans un modèle de ferromagnétisme. *Comm. Partial Differential Equations*, 27(7-8):1467–1495, 2002.

[CFKG03]   Rémi Carles, Clotilde Fermanian Kammerer, and Isabelle Gallagher. On the role of quadratic oscillations in nonlinear Schrödinger equations. *J. Funct. Anal.*, 203(2):453–493, 2003.

[CG01]   Andrea Corli and Olivier Guès. Stratified solutions for systems of conservation laws. *Trans. Amer. Math. Soc.*, 353(6):2459–2486, 2001.

[CGK02]   Thierry Colin, Cédric Galusinski, and Hans G. Kaper. Waves in ferromagnetic media. *Comm. Partial Differential Equations*, 27(7-8):1625–1658, 2002.

[CGL05]   Thierry Colin, Gérard Gallice, and Karen Laurioux. Intermediate models in nonlinear optics. *SIAM J. Math. Anal.*, 36(5):1664–1688, 2005.

[CGM03]   Christophe Cheverry, Olivier Guès, and Guy Métivier. Oscillations fortes sur un champ linéairement dégénéré. *Ann. Sci. École Norm. Sup. (4)*, 36(5):691–745, 2003.

[CGM04]   C. Cheverry, O. Guès, and G. Métivier. Large-amplitude high-frequency waves for quasilinear hyperbolic systems. *Adv. Differential Equations*, 9(7-8):829–890, 2004.

[Che96a]   Christophe Cheverry. Oscillations de faible amplitude pour les systèmes 2×2 de lois de conservation. *Asymptotic Anal.*, 12(1):1–24, 1996.

[Che96b]   Christophe Cheverry. Propagation d'oscillations près d'un point diffractif. *J. Math. Pures Appl. (9)*, 75(5):419–467, 1996.

[Che97]   Christophe Cheverry. Justification de l'optique géométrique non linéaire pour un système de lois de conservation. *Duke Math. J.*, 87(2):213–263, 1997.

[Che04]   Christophe Cheverry. Propagation of oscillations in real vanishing viscosity limit. *Comm. Math. Phys.*, 247(3):655–695, 2004.

[Chear]   Christophe Cheverry. Cascade of phases in turbulent flows. *Bull. SMF*, to appear.

[Chi91]   Jamel Chikhi. Sur la réflexion des oscillations pour un système à deux vitesses. *C. R. Acad. Sci. Paris Sér. I Math.*, 313(10):675–678, 1991.

[CKar]   Rémi Carles and Sahbi Keraani. On the role of quadratic oscillations in nonlinear Schrödinger equations II. The l^2-critical case. *Trans. AMS*, to appear.

[CL01]   Thierry Colin and David Lannes. Long-wave short-wave resonance for nonlinear geometric optics. *Duke Math. J.*, 107(2):351–419, 2001.

[CL03a]   Rémi Carles and David Lannes. Focusing at a point with caustic crossing for a class of nonlinear equations. *Colloque franco-tunisien, Hammamet*, 2003.

[CL03b]   Rémi Carles and David Lannes. Focusing of a pulse with arbitrary phase shift for a nonlinear wave equation. *Bull. Soc. Math. France*, 131(2):289–306, 2003.

[CL04]   Thierry Colin and David Lannes. Justification of and long-wave correction to Davey-Stewartson systems from quadratic hyperbolic systems. *Discrete Contin. Dyn. Syst.*, 11(1):83–100, 2004.

[CM04]   Rémi Carles and Luc Miller.   Semiclassical nonlinear
           Schrödinger equations with potential and focusing initial data.
           Osaka J. Math., 41(3):693–725, 2004.

[CMSar]   Rémi Carles, Norbert J. Mauser, and Hans P. Stimming.
           (semi)classical limit of the Hartree equation with harmonic po-
           tential. *SIAM J. Appl. Math.*, to appear.

[CN04a]   Rémi Carles and Yoshihisa Nakamura. Nonlinear Schrödinger
           equations with Stark potential. *Hokkaido Math. J.*, 33(3):719–
           729, 2004.

[CN04b]   Thierry Colin and Boniface Nkonga. A numerical model for
           light interaction with a two-level atom medium. *Phys. D*, 188(1-
           2):92–118, 2004.

[CN05]   Thierry Colin and Boniface Nkonga.   Multiscale numerical
           method for nonlinear Maxwell equations. *Discrete Contin. Dyn.
           Syst. Ser. B*, 5(3):631–658, 2005.

[Col02]   Thierry Colin. Rigorous derivation of the nonlinear Schrödinger
           equation and Davey-Stewartson systems from quadratic hyper-
           bolic systems. *Asymptot. Anal.*, 31(1):69–91, 2002.

[Cor95]   Andrea Corli. Weakly non-linear geometric optics for hyper-
           bolic systems of conservation laws with shock waves. *Asymp-
           totic Anal.*, 10(2):117–172, 1995.

[CR02]   Rémi Carles and Jeffrey Rauch. Focusing of spherical nonlinear
           pulses in $\mathbb{R}^{1+3}$. *Proc. Amer. Math. Soc.*, 130(3):791–804, 2002.

[CR04a]   Rémi Carles and Jeffrey Rauch. Focusing of spherical nonlinear
           pulses in $\mathbb{R}^{1+3}$. II. Nonlinear caustic. *Rev. Mat. Iberoamericana*,
           20(3):815–864, 2004.

[CR04b]   Rémi Carles and Jeffrey Rauch. Focusing of spherical nonlinear
           pulses in R^{1+3}. III. Sub and supercritical cases. *Tohoku Math.
           J. (2)*, 56(3):393–410, 2004.

[CT05]   Thierry Colin and Vincent Torri. Numerical scheme for the 2D
           Maxwell-Bloch equations modeling ultrashort pulses. *Preprint*,
           2005.

[DBR03]   Stephan De Bièvre and Didier Robert. Semiclassical propagation on $|\log \hbar|$ time scales. *Int. Math. Res. Not.*, (12):667–696, 2003.

[DJMR96]   Philippe Donnat, Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. Diffractive nonlinear geometric optics. In *Séminaire sur les Équations aux Dérivées Partielles, 1995–1996*, Sémin. Équ. Dériv. Partielles, pages Exp. No. XVII, 25. École Polytech., Palaiseau, 1996.

[Don94]   Philippe Donnat. Quelques contributions mathématiques en optique non linéaire, 1994. Thèse de l'École Polytechnique.

[Dor97]   Emmanuel Dormy. Modélisation dynamique de la dynamo terrestre, 1997. Thèse de l'Institut de Physique du Globe, Paris.

[DR97a]   Phillipe Donnat and Jeffrey Rauch. Dispersive nonlinear geometric optics. *J. Math. Phys.*, 38(3):1484–1523, 1997.

[DR97b]   Phillipe Donnat and Jeffrey Rauch. Modeling the dispersion of light. In *Singularities and oscillations (Minneapolis, MN, 1994/1995)*, volume 91 of *IMA Vol. Math. Appl.*, pages 17–35. Springer, New York, 1997.

[Dui74]   Johannes J. Duistermaat. Oscillatory integrals, Lagrange immersions and unfolding of singularities. *Comm. Pure Appl. Math.*, 27:207–281, 1974.

[Dum02]   Éric Dumas. Propagation of oscillations near a diffractive point for a dissipative and semilinear Klein-Gordon equation. *Comm. Partial Differential Equations*, 27(5-6):953–978, 2002.

[Dum03a]   Éric Dumas. An example of catastrophic self-focusing in nonlinear optics? *SIAM J. Math. Anal.*, 35(1):268–278, 2003.

[Dum03b]   Éric Dumas. Periodic multiphase nonlinear diffractive optics with curved phases. *Indiana Univ. Math. J.*, 52(3):769–810, 2003.

[Dum04]   Éric Dumas. Nonlinear diffractive optics with curved phases: beam dispersion and transition between light and shadow. *Asymptot. Anal.*, 38(1):47–91, 2004.

[Fri76]   F. G. Friedlander. The wave front set of the solution of a simple initial-boundary value problem with glancing rays. *Math. Proc. Cambridge Philos. Soc.*, 79(1):145–159, 1976.

[Gal98]   Isabelle Gallagher. Applications of Schochet's methods to parabolic equations. *J. Math. Pures Appl. (9)*, 77(10):989–1054, 1998.

[Gér93]   Patrick Gérard. Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire. In *Séminaire sur les Équations aux Dérivées Partielles, 1992–1993*, pages Exp. No. XIII, 13. École Polytech., Palaiseau, 1993.

[GG98]   Emmanuel Grenier and Olivier Guès. Boundary layers for viscous perturbations of noncharacteristic quasilinear hyperbolic problems. *J. Differential Equations*, 143(1):110–146, 1998.

[GM97]   Emmanuel Grenier and Nader Masmoudi. Ekman layers of rotating fluids, the case of well prepared initial data. *Comm. Partial Differential Equations*, 22(5-6):953–975, 1997.

[GMWZ04a]   Olivier Guès, Guy Métivier, Mark Williams, and Kevin Zumbrun. Boundary layer and long time stability for multidimensional viscous shocks. *Discrete Contin. Dyn. Syst.*, 11(1):131–160, 2004.

[GMWZ04b]   Olivier Guès, Guy Métivier, Mark Williams, and Kevin Zumbrun. Multidimensional viscous shocks. II. The small viscosity limit. *Comm. Pure Appl. Math.*, 57(2):141–218, 2004.

[GMWZ05a]   Olivier Guès, Guy Métivier, Mark Williams, and Kevin Zumbrun. Existence and stability of multidimensional shock fronts in the vanishing viscosity limit. *Arch. Ration. Mech. Anal.*, 175(2):151–244, 2005.

[GMWZ05b]   Olivier Guès, Guy Métivier, Mark Williams, and Kevin Zumbrun. Multidimensional viscous shocks. I. Degenerate symmetrizers and long time stability. *J. Amer. Math. Soc.*, 18(1):61–120, 2005.

[Gos05]   Laurent Gosse. A case study on the reliability of multiphase wkb approximation for the one-dimensional schrdinger equation. *Preprint HYKE2005-037*, 2005.

[Gre80]   Harvey Philip Greenspan. *The theory of rotating fluids*. Cambridge University Press, Cambridge, 1980. Reprint of the 1968 original, Cambridge Monographs on Mechanics and Applied Mathematics.

[Gre97a]   Emmanuel Grenier. Boundary layers for parabolic regularizations of totally characteristic quasilinear parabolic equations. *J. Math. Pures Appl. (9)*, 76(10):965–990, 1997.

[Gre97b]   Emmanuel Grenier. Oscillatory perturbations of the Navier-Stokes equations. *J. Math. Pures Appl. (9)*, 76(6):477–498, 1997.

[Gre98]   Emmanuel Grenier. Semiclassical limit of the nonlinear Schrödinger equation in small time. *Proc. Amer. Math. Soc.*, 126(2):523–530, 1998.

[Guè93]   Olivier Guès. Développement asymptotique de solutions exactes de systèmes hyperboliques quasilinéaires. *Asymptotic Anal.*, 6(3):241–269, 1993.

[Guè95a]   Olivier Guès. Ondes solitaires engendrées par interaction d'ondes oscillantes non linéaires. *J. Math. Pures Appl. (9)*, 74(3):199–252, 1995.

[Guè95b]   Olivier Guès. Perturbations visqueuses de problèmes mixtes hyperboliques et couches limites. *Ann. Inst. Fourier (Grenoble)*, 45(4):973–1006, 1995.

[GV03a]   David Gérard-Varet. A geometric optics type approach to fluid boundary layers. *Comm. Partial Differential Equations*, 28(9-10):1605–1626, 2003.

[GV03b]   David Gérard-Varet. Highly rotating fluids in rough domains. *J. Math. Pures Appl. (9)*, 82(11):1453–1498, 2003.

[GV05]   David Gérard-Varet. Formal derivation of boundary layers in fluid mechanics. *J. Math. Fluid Mech.*, 7:179–200, 2005.

[Hei93]   Arnaud Heibig. Error estimates for oscillatory solutions to hyperbolic systems of conservation laws. *Comm. Partial Differential Equations*, 18(1-2):281–304, 1993.

[HK83]   John K. Hunter and Joseph B. Keller. Weakly nonlinear high frequency waves. *Comm. Pure Appl. Math.*, 36(5):547–569, 1983.

[HK87]   John K. Hunter and Joseph B. Keller. Caustics of nonlinear waves. *Wave Motion*, 9(5):429–443, 1987.

[HMR86]   John K. Hunter, Andrew Majda, and Rodolfo R. Rosales. Resonantly interacting, weakly nonlinear hyperbolic waves. II. Several space variables. *Stud. Appl. Math.*, 75(3):187–226, 1986.

[Hun88]   John K. Hunter. Transverse diffraction of nonlinear waves and singular rays. *SIAM J. Appl. Math.*, 48(1):1–37, 1988.

[Hun95]   John K. Hunter. Asymptotic equations for nonlinear hyperbolic waves. In *Surveys in applied mathematics, Vol. 2*, volume 2 of *Surveys Appl. Math.*, pages 167–276. Plenum Press, New York, 1995.

[Jea02]   Pierre-Yves Jeanne. Optique géométrique pour des systèmes semi-linéaires avec invariance de jauge. *Mém. Soc. Math. Fr. (N.S.)*, 90:160, 2002.

[JMR93a]   Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves. *Duke Math. J.*, 70(2):373–404, 1993.

[JMR93b]   Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. Resonant one-dimensional nonlinear geometric optics. *J. Funct. Anal.*, 114(1):106–231, 1993.

[JMR94]   Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. Coherent nonlinear waves and the Wiener algebra. *Ann. Inst. Fourier (Grenoble)*, 44(1):167–196, 1994.

[JMR95a]   Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. Coherent and focusing multidimensional nonlinear geometric optics. *Ann. Sci. École Norm. Sup. (4)*, 28(1):51–113, 1995.

[JMR95b]   Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. Focusing at a point and absorption of nonlinear oscillations. *Trans. Amer. Math. Soc.*, 347(10):3921–3969, 1995.

[JMR95c]   Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. Trilinear compensated compactness and nonlinear geometric optics. *Ann. of Math. (2)*, 142(1):121–169, 1995.

[JMR96]   Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. Nonlinear oscillations beyond caustics. *Comm. Pure Appl. Math.*, 49(5):443–527, 1996.

[JMR98]   Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. Diffractive nonlinear geometric optics with rectification. *Indiana Univ. Math. J.*, 47(4):1167–1241, 1998.

[JMR99]   Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. Recent results in non-linear geometric optics. In *Hyperbolic problems: theory, numerics, applications, Vol. II (Zürich, 1998)*, volume 130 of *Internat. Ser. Numer. Math.*, pages 723–736. Birkhäuser, Basel, 1999.

[JMR00a]   Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. Caustics for dissipative semilinear oscillations. *Mem. Amer. Math. Soc.*, 144(685):viii+72, 2000.

[JMR00b]   Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. Transparent nonlinear geometric optics and Maxwell-Bloch equations. *J. Differential Equations*, 166(1):175–250, 2000.

[Jol83]   Jean-Luc Joly. Sur la propagation des oscillations par un système hyperbolique semi-linéaire en dimension 1 d'espace. *C. R. Acad. Sci. Paris Sér. I Math.*, 296(15):669–672, 1983.

[JR91]   Jean-Luc Joly and Jeffrey Rauch. Nonlinear resonance can create dense oscillations. In *Microlocal analysis and nonlinear*

waves (Minneapolis, MN, 1988–1989), volume 30 of *IMA Vol. Math. Appl.*, pages 113–123. Springer, New York, 1991.

[JR92]   Jean-Luc Joly and Jeffrey Rauch. Justification of multidimensional single phase semilinear geometric optics. *Trans. Amer. Math. Soc.*, 330(2):599–623, 1992.

[Kal89]   Leonid A. Kalyakin. Long-wave asymptotics. Integrable equations as the asymptotic limit of nonlinear systems. *Uspekhi Mat. Nauk*, 44(1(265)):5–34, 247, 1989.

[Kle00]   Rupert Klein. Asymptotic analyses for atmospheric flows and the construction of asymptotically adaptive numerical methods. *ZAMM Z. Angew. Math. Mech.*, 80(11-12):765–777, 2000. Special issue on the occasion of the 125th anniversary of the birth of Ludwig Prandtl.

[KM81]   Sergiu Klainerman and Andrew Majda. Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Comm. Pure Appl. Math.*, 34(4):481–524, 1981.

[KM82]   Sergiu Klainerman and Andrew Majda. Compressible and incompressible fluids. *Comm. Pure Appl. Math.*, 35(5):629–651, 1982.

[Lan98]   David Lannes. Dispersive effects for nonlinear geometrical optics with rectification. *Asymptot. Anal.*, 18(1-2):111–146, 1998.

[Lan03a]   David Lannes. Consistency of the KP approximation. *Discrete Contin. Dyn. Syst.*, (suppl.):517–525, 2003. Dynamical systems and differential equations (Wilmington, NC, 2002).

[Lan03b]   David Lannes. Secular growth estimates for hyperbolic systems. *J. Differential Equations*, 190(2):466–503, 2003.

[Lax57]   Peter D. Lax. Asymptotic solutions of oscillatory initial value problems. *Duke Math. J.*, 24:627–646, 1957.

[Leb96]   Hervé Leblond. Electromagnetic waves in ferrites: from linear absorption to the nonlinear Schrödinger equation. *J. Phys. A*, 29(15):4623–4639, 1996.

[Leb99]   Hervé Leblond.   Electromagnetic waves in ferromagnets: a
            Davey-Stewartson-type model. *J. Phys. A*, 32(45):7907–7932,
            1999.

[Leb01]   Hervé Leblond.   Rigorous derivation of the NLS in magnetic
            films. *J. Phys. A*, 34(45):9687–9712, 2001.

[Leb02]   Hervé Leblond. Mixed perturbative expansion: the validity of
            a model for the cascading. *J. Nonlinear Math. Phys.*, 9(2):126–
            143, 2002.

[LL69]   Lev D. Landau and Evgenij M. Lifshitz. *Électrodynamique des
            milieux continus*. Physique théorique (008). Mir, New York,
            1969.

[LP04]   Stéphane Labbé and Lionel Paumond. Numerical comparisons
            of two long-wave limit models. *M2AN Math. Model. Numer.
            Anal.*, 38(3):419–436, 2004.

[LR01]   David Lannes and Jeffrey Rauch.   Validity of nonlinear geo-
            metric optics with times growing logarithmically. *Proc. Amer.
            Math. Soc.*, 129(4):1087–1096, 2001.

[Lud66]   Donald Ludwig. Uniform asymptotic expansions at a caustic.
            Comm. Pure Appl. Math., 19:215–250, 1966.

[MA88]   Andrew J. Majda and Miguel Artola. Nonlinear geometric op-
            tics for hyperbolic mixed problems. In *Analyse mathématique
            et applications*, pages 319–356. Gauthier-Villars, Montrouge,
            1988.

[Maj84]   Andrew Majda. *Compressible fluid flow and systems of con-
            servation laws in several space variables*, volume 53 of *Applied
            Mathematical Sciences*. Springer-Verlag, New York, 1984.

[Mas00]   Nader Masmoudi. Ekman layers of rotating fluids: the case of
            general initial data. *Comm. Pure Appl. Math.*, 53(4):432–483,
            2000.

[Mel75a]   Richard B. Melrose.   Local Fourier-Airy integral operators.
            Duke Math. J., 42(4):583–604, 1975.

[Mel75b]   Richard B. Melrose. Microlocal parametrices for diffractive boundary value problems. *Duke Math. J.*, 42(4):605–635, 1975.

[MPT85]   David McLaughlin, George Papanicolaou, and Luc Tartar. Weak limits of semilinear hyperbolic systems with oscillating data. In *Macroscopic modelling of turbulent flows (Nice, 1984)*, volume 230 of *Lecture Notes in Phys.*, pages 277–289. Springer, Berlin, 1985.

[MR84]   Andrew Majda and Rodolfo Rosales. Resonantly interacting weakly nonlinear hyperbolic waves. I. A single space variable. *Stud. Appl. Math.*, 71(2):149–179, 1984.

[MR03]   Guy Métivier and Jeffrey Rauch. Real and complex regularity are equivalent for hyperbolic characteristic varieties. *Differential Integral Equations*, 16(8):993–999, 2003.

[MS98]   Guy Métivier and Steven Schochet. Trilinear resonant interactions of semilinear hyperbolic waves. *Duke Math. J.*, 95(2):241–304, 1998.

[MS01]   Guy Métivier and Steven Schochet. The incompressible limit of the non-isentropic Euler equations. *Arch. Ration. Mech. Anal.*, 158(1):61–90, 2001.

[Mus04]   Alexis Museux. Stratified weak solutions of the 1-D Lagrangian Euler equations are viscosity solutions. *Adv. Differential Equations*, 9(11-12):1395–1436, 2004.

[NM92]   Alan C. Newell and Jerome V. Moloney. *Nonlinear optics.* Advanced Topics in the Interdisciplinary Mathematical Sciences. Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1992.

[Ped79]   J. Pedlovsky. *Geophysical fluid dynamics.* Springer Verlag, Berlin, 1979.

[Peg88]   Robert L. Pego. Some explicit resonating waves in weakly nonlinear gas dynamics. *Stud. Appl. Math.*, 79(3):263–270, 1988.

[Pen92]   Yue-Jun Peng. Solutions faibles globales pour l'équation d'Euler d'un fluide compressible avec de grandes données initiales. *Comm. Partial Differential Equations*, 17(1-2):161–187, 1992.

[Rau96]   Jeffrey Rauch. *Lectures on Geometric Optics.* http://www.math.lsa.umich.edu/~rauch/oldnlgonotes.pdf, 1996.

[RR88]   Jeffrey Rauch and Michael Reed. Bounded, stratified and striated solutions of hyperbolic systems. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. IX (Paris, 1985–1986)*, volume 181 of *Pitman Res. Notes Math. Ser.*, pages 334–351. Longman Sci. Tech., Harlow, 1988.

[San02]   David Sanchez. Boundary layer on a high-conductivity domain. *Commun. Pure Appl. Anal.*, 1(4):547–564, 2002.

[San05]   David Sanchez. Long waves in ferromagnetic media, Khokhlov-Zabolotskaya equation. *J. Differential Equations*, 210(2):263–289, 2005.

[Sch94a]   Steven Schochet. Fast singular limits of hyperbolic PDEs. *J. Differential Equations*, 114(2):476–512, 1994.

[Sch94b]   Steven Schochet. Resonant nonlinear geometric optics for weak solutions of conservation laws. *J. Differential Equations*, 113(2):473–504, 1994.

[Sch98a]   Guido Schneider. Approximation of the Korteweg-de Vries equation by the nonlinear Schrödinger equation. *J. Differential Equations*, 147(2):333–354, 1998.

[Sch98b]   Guido Schneider. Justification of modulation equations for hyperbolic systems via normal forms. *NoDEA Nonlinear Differential Equations Appl.*, 5(1):69–82, 1998.

[Sch98c]   Guido Schneider. The long wave limit for a Boussinesq equation. *SIAM J. Appl. Math.*, 58(4):1237–1245, 1998.

[Ser95]   Denis Serre. Oscillations non-linéaires hyperboliques de grande amplitude; dim$\geq$ 2. In *Nonlinear variational problems and partial differential equations*, pages 245–294. Harlow, Longman Sci. Tech., Isola d'Elba (1990), 1995.

[SS99]   Catherine Sulem and Pierre-Louis Sulem. *The nonlinear Schrödinger equation. Self-focusing and wave collapse*, volume 139 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999.

[ST96]   Monique Sablé-Tougeron. Justification de l'optique géométrique faiblement non linéaire pour le problème mixte: cas des concentrations. *Asymptotic Anal.*, 12(3):169–186, 1996.

[SU03]   Guido Schneider and Hannes Uecker. Existence and stability of modulating pulse solutions in Maxwell's equations describing nonlinear optics. *Z. Angew. Math. Phys.*, 54(4):677–712, 2003.

[Sue05a]   Franck Sueur. Entropy boundary layers. *Preprint*, 2005.

[Sue05b]   Franck Sueur. Remarks about a theorem by J. Rauch. *Indiana University Mathematics Journal*, 54(4):1107–1144, 2005.

[Sueara]   Franck Sueur. Couches limites semilinéaires. *Annales de la Faculté des Sciences de Toulouse*, to appear.

[Suearb]   Franck Sueur. Couches limites: un problème inverse. *Communications in Partial Differential Equations*, to appear.

[Suearc]   Franck Sueur. Perturbations visqueuses de solutions discontinues de systèmes hyperboliques semilinéaires. *Annales de l'Institut Fourier*, to appear.

[SW00]   Guido Schneider and C. Eugene Wayne. The long-wave limit for the water wave problem. I. The case of zero surface tension. *Comm. Pure Appl. Math.*, 53(12):1475–1535, 2000.

[SW02]   Guido Schneider and C. Eugene Wayne. The rigorous approximation of long-wavelength capillary-gravity waves. *Arch. Ration. Mech. Anal.*, 162(3):247–285, 2002.

[Tar81]   Luc Tartar. Solutions oscillantes des équations de Carleman. In *Goulaouic-Meyer-Schwartz Seminar, 1980–1981*, pages Exp. No. XII, 15. École Polytech., Palaiseau, 1981.

[Tar84]   Luc Tartar. Étude des oscillations dans les équations aux dérivées partielles non linéaires. In *Trends and applications of pure mathematics to mechanics (Palaiseau, 1983)*, volume 195 of *Lecture Notes in Phys.*, pages 384–412. Springer, Berlin, 1984.

[Tex04]   Benjamin Texier. The short-wave limit for nonlinear symmetric hyperbolic systems. *Adv. Differential Equations*, 9(1-2):1–52, 2004.

[Tex05]   Benjamin Texier. WKB asymptotics for the Euler-Maxwell equations. *Asymptot. Anal.*, 42(3-4):211–250, 2005.

[Whi99]   Gerald B. Whitham. *Linear and nonlinear waves*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1999. Reprint of the 1974 original, A Wiley-Interscience Publication.

[Wil93]   Mark Williams. Resonant reflection of multidimensional semilinear oscillations. *Comm. Partial Differential Equations*, 18(11):1901–1959, 1993.

[Wil99]   Mark Williams. Highly oscillatory multidimensional shocks. *Comm. Pure Appl. Math.*, 52(2):129–192, 1999.

[Wil00]   Mark Williams. Boundary layers and glancing blow-up in nonlinear geometric optics. *Ann. Sci. École Norm. Sup. (4)*, 33(3):383–432, 2000.